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J Sau

Institut de Physique Nucléaire (and IN2P3), Université Claude Bernard Lyon I, 43, Bd du 11 Novembre 1918, 69621 Villeurbanne, France

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Abstract. The total energy of a system of non-interacting particles is explicitly given as a function of the total angular momentum of the system, J. Such a representation is also achieved when two-body interactions of central force type are present. Comparison with the rigid body case leads to interesting conclusions concerning the potential energy of the system.

1. Introduction

The manifestations of collective rotational motion appear frequently in the study of physical systems of particles. Usually *ad hoc* phenomenological Hamiltonians are employed in these cases. However, for systems of interacting particles, it will be interesting to understand the occurrence of collective rotational motion without assuming it from the start, as in the usual phenomenological method, especially as the same system can present aspects where the particles seem to move independently (for instance, in nuclear physics).

The collective variable associated with rotational motion is obviously the angular momentum around the centre of mass, $\boldsymbol{J}$. Since two components of $\boldsymbol{J}$ cannot belong to the same canonical set (Goldstein 1964), we must choose J, the norm of $\boldsymbol{J}(J = (J_x^2 + J_y^2 + J_z^2)^{1/2})$, and one of its components, say J_z. The conjugate momenta of J and J_z in classical and quantum mechanics can be found elsewhere (Sau 1978). Then, in theory, the explicit expansion of a rotationally invariant Hamiltonian in powers of J can be given following, for instance, a method outlined by Villars (1965) and Rowe (1967).

Let us first consider the simple case of two spinless particles with central interaction. The 'collective' rotational variable is the norm L of the orbital momentum around the centre of mass, and the Hamiltonian in the centre of mass takes the form

$$H = A + BL^2. \tag{1}$$

Here A and B do not depend on L but on 'intrinsic' variables (here r and p_r). Now, if a set of states is such that $\langle A \rangle$ and $\langle B \rangle$ remain approximately constant, one says that they form a rotational band with the usual $l(l+1)$ pattern.

The aim of this paper is to generalise an expansion of the type (1), where the dependence of H on J is explicitly given, to many-particle systems. This problem is solved here in classical mechanics. The case in which the Hamiltonian is the total kinetic energy is considered, and the presence of a potential energy of the kind

$$V = \sum_{i>j} V_{ij}(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$$

is then examined. Obviously we cannot claim that the quantum result would be obtained by simple quantisation of the classical one. However, apart from the relative ease of formulation, the classical derivation is interesting since it has been shown (Sau 1978) that, for sufficiently large angular momentum, the quantum conjugate variable of J approximates the classical one; then one may expect that the quantisation of the classical solution will give the quantum one approximately, at least for large angular momentum. Our basic tool here will be the classical Poisson bracket (CPB), denoted by {,}. Our approach, however, will be different from that of Villars (1965) and Rowe (1967), for, as we shall see, an expansion in powers of J does not converge in general.

2. The kinetic energy expansion

2.1. *The kinetic energy*

We start from the kinetic energy of N particles in the centre of mass,

$$H=\sum_{i=1}^{N}\frac{P_i^2}{2m_i},$$

with

$$\sum_{i=1}^{N}\boldsymbol{P}_i=0,\qquad \sum_{i=1}^{N}m_i\boldsymbol{r}_i=0.$$

Here $\boldsymbol{r}_i$ is the position vector of particle i relative to the centre of mass G, and $\boldsymbol{P}_i$ is its linear momentum. G will be the centre of a Cartesian system with unitary vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, the directions of which are fixed in space. Let us call $\boldsymbol{J}$ the total angular momentum around the centre of mass, given by

$$\boldsymbol{J}=\sum_{i=1}^{N}\boldsymbol{r}_i\wedge\boldsymbol{P}_i.$$

We then obtain the CPBS

$$\{(\boldsymbol{r}_i\cdot\boldsymbol{e}_k),(\boldsymbol{P}_j\cdot\boldsymbol{e}_l)\}=(\delta_{ij}-m_j/M)\delta_{kl},\qquad \{\boldsymbol{r}_i,\boldsymbol{J}\cdot\boldsymbol{n}\}=\boldsymbol{n}\wedge\boldsymbol{r}_i,\qquad \{\boldsymbol{P}_i,\boldsymbol{J}\cdot\boldsymbol{n}\}=\boldsymbol{n}\wedge\boldsymbol{P}_i, \tag{2}$$

where $\boldsymbol{n}$ is any constant vector and M is the total mass.

In the following we shall need CPBS of the kind $\{F(\boldsymbol{r}_j),\boldsymbol{P}_i\}$, where F is a function of the $\boldsymbol{r}_j$, $j=1,2,\ldots,N$. Owing to the constraint on the $\boldsymbol{r}_j$, F is actually a function of $(N-1)\boldsymbol{r}_j$ only. As the result is independent of the chosen $\boldsymbol{r}_j$, we shall take the first $N-1$ throughout this paper. The CPBS can be evaluated with the help of the first of the relations (2) as

$$\{F,P_{ix}\}=\sum_{j=1}^{N-1}\frac{\partial F}{\partial x_j}\left(\delta_{ij}-\frac{m_i}{M}\right).$$

Since similar expressions are obtained for P_{iy} and P_{iz}, we may write

$$\{F,\boldsymbol{P}_i\}=\sum_{j=1}^{N-1}\boldsymbol{\nabla}_jF\left(\delta_{ij}-\frac{m_i}{M}\right). \tag{3}$$

2.2. *The canonical set*

We recall now the canonical set of variables connected with the rotational problem (Sau 1978). Let $\boldsymbol{u}_3$ be a unitary vector, which we shall call a vector attached to the many-body system. The vector $\boldsymbol{u}_3$ is a function of the $\boldsymbol{r}_i$ and satisfies

$$\{\boldsymbol{u}_3, \boldsymbol{J}\,.\,\boldsymbol{n}\} = \boldsymbol{n} \wedge \boldsymbol{u}_3. \tag{4}$$

We note that $\boldsymbol{u}_3$ can be a particular $\boldsymbol{r}_i/|\boldsymbol{r}_i|$ or can be defined more symmetrically (a principal direction of the inertial tensor, for instance).

We define now the projection $\boldsymbol{\rho}$ of $\boldsymbol{u}_3$ on the plane orthogonal to $\boldsymbol{J}$, namely

$$\boldsymbol{\rho} = \boldsymbol{u}_3 - J_3\boldsymbol{J}/J^2, \tag{5}$$

with $J_3 = (\boldsymbol{J}\,.\,\boldsymbol{u}_3)$. Note that $\boldsymbol{\rho}\,.\,\boldsymbol{J} = 0$ and $\{\boldsymbol{\rho}, \boldsymbol{J}\,.\,\boldsymbol{n}\} = \boldsymbol{n}\wedge\boldsymbol{\rho}$. To complete our notation we use X, Y, Z for the projections of ρ on $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ respectively, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ for the unitary vectors which complete the Cartesian frame attached to the many-body system (with $\boldsymbol{u}_1\wedge\boldsymbol{u}_2 = \boldsymbol{u}_3$ and cyclic permutation), and J_1, J_2, J_3 for the projections of $\boldsymbol{J}$ on these axes.

It has been shown (Sau 1978) that the three pairs of conjugate momenta which form the canonical set for the description of rotational motion in classical mechanics are (J, α), (J_z, β), (J_3, γ), with

$$\alpha = \tan^{-1}[ZJ/(YJ_x - XJ_y)], \qquad \beta = \tan^{-1}(J_y/J_x), \qquad \gamma = \tan^{-1}(J_1/J_2). \tag{6}$$

Indeed we get $\{\alpha, J\} = 1$, $\{\beta, J_z\} = 1$, $\{\gamma, J_3\} = 1$, with all other CPBs being zero, and the necessary relations for a canonical set are then fulfilled.

Since H is rotationally invariant, one can see immediately that $\{\beta, H\} = \{J_z, H\} = 0$, and thus β and J_z are both cyclic coordinates. This fact simply shows the well-known degeneracy of a rotationally invariant Hamiltonian. Note that α is also cyclic, since $\{J, H\} = 0$.

Apart from the three pairs above, other variables (usually called intrinsic) are obviously necessary for completing the canonical set. The derivation which follows does not depend on them, however.

2.3. *The J dependence of H*

The Hamiltonian H is a function of all the variables of the canonical set. Our derivation uses the fact that a CPB like $\{\alpha, H\}$ is equivalent to the partial derivative $\partial H/\partial J$.

In the following we shall make an extensive use of the relation (A5) derived in the Appendix. With $\boldsymbol{v}$ defined as $\boldsymbol{v} = \{\boldsymbol{u}_3, H\}$, relation (A5) leads to the differential equation for H

$$\partial H/\partial J = \{\alpha, H\} = [J/(J^2 - J_3^2)](\boldsymbol{u}_3\wedge\boldsymbol{v})\,.\,\boldsymbol{J}. \tag{7}$$

Taking account of relation (3) we obtain the expression for $\boldsymbol{v}$ in component form as

$$v_x = \sum_{i=1}^{N-1}\left(\frac{\partial u_{3_x}}{\partial x_i}P_{i_x} + \frac{\partial u_{3_x}}{\partial y_i}P_{i_y} + \frac{\partial u_{3_x}}{\partial z_i}P_{i_z}\right)\frac{1}{m_i} \tag{8}$$

or more compactly as

$$\boldsymbol{v} = \sum_{i=1}^{N-1}\frac{1}{m_i}(\boldsymbol{P}_i\,.\,\boldsymbol{\nabla}_i)\boldsymbol{u}_3. \tag{8$'$}$$

Let us now deal with $S=(\boldsymbol{u}_3\wedge\boldsymbol{v})\,.\,\boldsymbol{J}$. Defining $\boldsymbol{w}$ as $\boldsymbol{w}=\{\boldsymbol{u}_3, S\}$, we obtain

$$\partial S/\partial J=\{\alpha, S\}=[J/(J^2-J_3^2)](\boldsymbol{u}_3\wedge\boldsymbol{w})\,.\,\boldsymbol{J}. \tag{9}$$

The expression for $\boldsymbol{w}$ is

$$\boldsymbol{w}=\{\boldsymbol{u}_3,(\boldsymbol{u}_3\wedge\boldsymbol{v})\,.\,\boldsymbol{J}\}=(\boldsymbol{u}_3\wedge\boldsymbol{v})\wedge\boldsymbol{u}_3+\{\boldsymbol{u}_3,(\boldsymbol{u}_3\wedge\boldsymbol{v})\}\,.\,\boldsymbol{J}. \tag{10}$$

Since the $\boldsymbol{P}_i$ are present in the expression for $\boldsymbol{v}$, the CPB above can be evaluated using relation (3). This evaluation gives rise to a 3×3 matrix A, such that

$$A_{xz}=\sum_{j=1\,k=1}^{N-1}\left(\frac{\partial u_{3_x}}{\partial x_k}\frac{\partial u_{3_z}}{\partial x_j}+\frac{\partial u_{3_x}}{\partial y_k}\frac{\partial u_{3_z}}{\partial y_j}+\frac{\partial u_{3_x}}{\partial z_k}\frac{\partial u_{3_z}}{\partial z_j}\right)\frac{1}{m_j}\left(\delta_{kj}-\frac{m_j}{M}\right)$$

or more compactly

$$A=\sum_{j=1\,k=1}^{N-1}(\boldsymbol{\nabla}_j\boldsymbol{u}_3)\,.\,(\boldsymbol{\nabla}_k\boldsymbol{u}_3)\frac{1}{m_j}\left(\delta_{kj}-\frac{m_j}{M}\right). \tag{11}$$

Here the scalar product is only between the $\boldsymbol{\nabla}$'s. Then the expression for $\boldsymbol{w}$ in (10) is

$$\boldsymbol{w}=-A\,.\,(\boldsymbol{u}_3\wedge\boldsymbol{J})+\boldsymbol{v}-\boldsymbol{u}_3(\boldsymbol{u}_3\,.\,\boldsymbol{v}),$$

and thus

$$\boldsymbol{J}\,.\,(\boldsymbol{u}_3\wedge\boldsymbol{w})=\boldsymbol{J}\,.\;B\,.\,\boldsymbol{J}+(\boldsymbol{u}_3\wedge\boldsymbol{v})\,.\,\boldsymbol{J}. \tag{12}$$

Here B is a 3×3 matrix, written formally as

$$B=-\boldsymbol{u}_3\wedge(A\wedge\boldsymbol{u}_3) \tag{13}$$

or in component form as

$$B_{xz}=-u_{3_y}(A_{zx}u_{3_y}-A_{zy}u_{3_x})+u_{3_z}(A_{yx}u_{3_y}-A_{yy}u_{3_x}),$$

and has the properties

$$B\,.\,\boldsymbol{u}_3=0,\qquad B^+=B\qquad\text{since}\quad A^+=A. \tag{14}$$

The differential equation for S then has the form

$$\partial S/\partial J=[J/(J^2-J_3^2)](\boldsymbol{J}\,.\,B\,.\,\boldsymbol{J}+S). \tag{15}$$

We now let $F=\boldsymbol{J}\,.\,\boldsymbol{B}\,.\,\boldsymbol{J}$. The problem for F is simpler, since the $\boldsymbol{P}_i$ do not occur in $\boldsymbol{B}$. Again using the relation (A5) we obtain

$$\partial F/\partial J=\{\alpha, F\}=[J/(J^2-J_3^2)](\boldsymbol{u}_3\wedge\boldsymbol{f})\,.\,\boldsymbol{J}$$

with $\boldsymbol{f}=\{\boldsymbol{u}_3, F\}$. Formally $\boldsymbol{f}=2(\boldsymbol{J}\,.\,B)\wedge\boldsymbol{u}_3$ or explicitly

$$f_x=2[(J_xB_{xy}+J_yB_{yy}+J_zB_{zy})u_{3_z}-(J_xB_{xz}+J_yB_{yz}+J_zB_{zz})u_{3_y}].$$

Then using (14) ($\boldsymbol{B}\boldsymbol{u}_3=0$),

$$\partial F/\partial J=[2J/(J^2-J_3^2)]F,$$

the solution of which is

$$F=\boldsymbol{J}\,.\,B\,.\,\boldsymbol{J}=C(J^2-J_3^2). \tag{16}$$

The result (16) can be used in (15), and the differential equation for S is now

$$\partial S/\partial J=[J/(J^2-J_3^2)]S+CJ,$$

the solution of which is

$$S = C(J^2 - J_3^2) + D(J^2 - J_3^2)^{1/2}. \tag{17}$$

The quantities D and C, which are integration constants, are independent of J. The result (17), used in (7), leads to

$$\partial H/\partial J = CJ + D[J/(J^2 - J_3^2)^{1/2}].$$

Therefore the definitive result for H is

$$H = CJ^2/2 + D(J^2 - J_3^2)^{1/2} + E,$$

where E is independent of J. In order to obtain a more homogeneous expression we shall write H as

$$H = \tfrac{1}{2}C(J^2 - J_3^2) + D(J^2 - J_3^2)^{1/2} + E. \tag{18}$$

The quantity $\frac{1}{2}CJ_3^2$, independent of J, can be added to E.

The relation (18) is exact and represents the most general expression for H written explicitly as a function of J. It generalises relation (1) of the two-body Hamiltonian.

The quantities C, D, E can be written by identification, i.e. successively

$$C = \frac{\boldsymbol{J}.\boldsymbol{B}.\boldsymbol{J}}{J^2 - J_3^2}, \qquad D = \frac{S - \boldsymbol{J}.\boldsymbol{B}.\boldsymbol{J}}{(J^2 - J_3^2)^{1/2}}, \qquad E = H + \frac{\boldsymbol{J}.\boldsymbol{B}.\boldsymbol{J}}{2} - S. \tag{19}$$

Since the 'intrinsic' variables are not specified, we can say nothing else about C, D, E. In (19) they are given as functions of the original $\boldsymbol{r}_i$; this would be an advantage for calculations involving independent particle wavefunctions, as is often the case in nuclear physics. Although J appears in (19), C, D, E are independent of J.

We can remark, from relation (18), that an expansion in powers of J does not converge in general, and this surely remains true in quantum mechanics. An expansion in powers of J^{-1} must be looked for instead.

Finally, if we add to the kinetic energy a potential energy V such that $\{\alpha, V\} = 0$, for instance of the kind $V = \Sigma_{i=1}^{N-1} V_{ij}(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$, this potential energy is simply added to E and the expression for H as a function of J remains that given by relation (18).

It seems normal that J, in (18), occurs at most with the power two, since H is quadratic in the P_i. Attempts to eliminate the $(J^2 - J_3^2)^{1/2}$ term (by cancellation of D) have failed up to now. Indeed, the quantities C, D, E depend only on the chosen system $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$, and it has not been possible to find a particular system where $D = 0$.

2.4. *Special choices for* u_3

Here we shall calculate the expressions of C and D (relations (19)) for two particular $\boldsymbol{u}_3$: first $\boldsymbol{u}_3$ is taken as the direction of a given particle, say $\boldsymbol{u}_3 = \boldsymbol{r}_1/|\boldsymbol{r}_1|$, and after as a principal direction of the inertial tensor, in connection with the problem of a rigid body.

2.4.1. $\boldsymbol{u}_3 = \boldsymbol{r}_1/|\boldsymbol{r}_1|$. With expression (11) we obtain A in dyadic notation as

$$A = (1/m_1 r_1^4)(1 - m_1/M)(r_1^2 I - \boldsymbol{r}_1\boldsymbol{r}_1),$$

where I is the unit dyadic. Relation (13) leads to $B = A$. Then

$$C = (1/m_1 r_1^2)(1 - m_1/M).$$

Using (8′) we find

$$S=(1/m_1r_1^2)\boldsymbol{j}_1\,.\,\boldsymbol{J},$$

where $\boldsymbol{j}_1=\boldsymbol{r}_1\wedge\boldsymbol{P}_1$. Finally

$$D=(1/m_1r_1^2)[\boldsymbol{j}_1\,.\,\boldsymbol{J}-(1-m_1/M)(J^2-J_3^2)](J^2-J_3^2)^{-1/2}.$$

2.4.2. $\boldsymbol{u}_3$ is a principal direction of the inertial tensor. Let the $\boldsymbol{u}_i$ be eigendirections of the quadrupole tensor Q; in dyadic notation

$$Q=\sum_{k=1}^{N}m_k\boldsymbol{r}_k\boldsymbol{r}_k,\qquad Q\,.\,\boldsymbol{u}_i=\lambda_i\boldsymbol{u}_i\qquad(i=1,2,3).\tag{20}$$

The principal momenta of inertia are related to the λ_i by

$$I_1=\lambda_2+\lambda_3,\qquad I_2=\lambda_1+\lambda_3,\qquad I_3=\lambda_1+\lambda_2.$$

In order to obtain A and B (relations (11) and (13)) we need the derivatives of the vector $\boldsymbol{u}_3$, like $\partial\boldsymbol{u}_3/\partial x_j\,(j=1,2,\ldots,N-1)$. Since $\boldsymbol{u}_3$ is a unitary vector, a derivative like $\partial\boldsymbol{u}_3/\partial\boldsymbol{x}_j$ is orthogonal to $\boldsymbol{u}_3$ and can then be expanded on $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. The coefficients of the expansion can be found using relations (20). We have

$$\frac{\partial Q}{\partial x_j}\,.\,\boldsymbol{u}_3+Q\,.\,\frac{\partial\boldsymbol{u}_3}{\partial x_j}=\frac{\partial\lambda_3}{\partial x_j}\boldsymbol{u}_3+\lambda_3\frac{\partial\boldsymbol{u}_3}{\partial\boldsymbol{x}_j}.$$

Taking the scalar product with $\boldsymbol{u}_1$, for instance, we find

$$\boldsymbol{u}_1\,.\left(\frac{\partial Q}{\partial x_j}\,.\,\boldsymbol{u}_3\right)+\lambda_1\boldsymbol{u}_1\,.\,\frac{\partial\boldsymbol{u}_3}{\partial x_j}=\lambda_3\boldsymbol{u}_1\,.\,\frac{\partial\boldsymbol{u}_3}{\partial x_j}.$$

The quantity $(\boldsymbol{u}_1\,.\,\partial\boldsymbol{u}_3/\partial x_j)$ is precisely the coefficient of $\boldsymbol{u}_1$ in the expansion of $\partial\boldsymbol{u}_3/\partial x_j$. We obtain $\partial Q/\partial x_j$ from expression (20) for Q,

$$\frac{\partial Q_{xx}}{\partial x_j}=2m_j(x_j-x_N),\qquad\frac{\partial Q_{xy}}{\partial x_j}=\frac{\partial Q_{yx}}{\partial x_j}=m_j(y_j-y_N),\qquad\frac{\partial Q_{xz}}{\partial x_j}=\frac{\partial Q_{zx}}{\partial x_j}=m_j(z_j-z_N),$$

the other derivatives being zero. Then

$$\left(\boldsymbol{u}_1\,.\,\frac{\partial\boldsymbol{u}_3}{\partial x_j}\right)=\frac{m_j}{\lambda_3-\lambda_1}[u_{1_x}((\boldsymbol{r}_j-\boldsymbol{r}_N)\,.\,\boldsymbol{u}_3)+u_{3_x}((\boldsymbol{r}_j-\boldsymbol{r}_N)\,.\,\boldsymbol{u}_1)].$$

A similar calculation can be made for $(\boldsymbol{u}_2\,.\,\partial\boldsymbol{u}_3/\partial x_j)$. Finally $\boldsymbol{\nabla}_j\boldsymbol{u}_3$ can be written in dyadic form as

$$\begin{aligned}\boldsymbol{\nabla}_j\boldsymbol{u}_3=&\frac{m_j}{\lambda_3-\lambda_1}[((\boldsymbol{r}_j-\boldsymbol{r}_N)\,.\,\boldsymbol{u}_3)\boldsymbol{u}_1\boldsymbol{u}_1+((\boldsymbol{r}_j-\boldsymbol{r}_N)\,.\,\boldsymbol{u}_1)\boldsymbol{u}_3\boldsymbol{u}_1]\\&+\frac{m_j}{\lambda_3-\lambda_2}[((\boldsymbol{r}_j-\boldsymbol{r}_N)\,.\,\boldsymbol{u}_3)\boldsymbol{u}_2\boldsymbol{u}_2+((\boldsymbol{r}_j-\boldsymbol{r}_N)\,.\,\boldsymbol{u}_2)\boldsymbol{u}_3\boldsymbol{u}_2].\end{aligned}\tag{21}$$

Relation (21) can be used in (11); taking account of $\Sigma_{j=1}^{N}m_j\boldsymbol{r}_j=0$ and of $\Sigma_{j=1}^{N}m_j\boldsymbol{r}_j(\boldsymbol{r}_j\,.\,\boldsymbol{u}_i)=\lambda_i\boldsymbol{u}_i$, we obtain A in dyadic notation as

$$A=\frac{\lambda_3+\lambda_1}{(\lambda_3-\lambda_1)^2}\boldsymbol{u}_1\boldsymbol{u}_1+\frac{\lambda_2+\lambda_3}{(\lambda_3-\lambda_2)^2}\boldsymbol{u}_2\boldsymbol{u}_2.$$

Then B, from (13), is given by

$$B=\frac{\lambda_3+\lambda_1}{(\lambda_3-\lambda_1)^2}\boldsymbol{u}_2\boldsymbol{u}_2+\frac{\lambda_2+\lambda_3}{(\lambda_3-\lambda_2)^2}\boldsymbol{u}_1\boldsymbol{u}_1.$$

Therefore

$$\boldsymbol{J}.\boldsymbol{B}.\boldsymbol{J}=\frac{\lambda_3+\lambda_1}{(\lambda_3-\lambda_1)^2}J_2^2+\frac{\lambda_2+\lambda_3}{(\lambda_3-\lambda_2)^2}J_1^2, \tag{22}$$

and C, defined by (19), is given by

$$C=\frac{1}{2}\left(\frac{\lambda_3+\lambda_1}{(\lambda_3-\lambda_1)^2}+\frac{\lambda_3+\lambda_2}{(\lambda_3-\lambda_2)^2}\right)+\frac{1}{2}\left(\frac{\lambda_3+\lambda_1}{(\lambda_3-\lambda_1)^2}-\frac{\lambda_3+\lambda_2}{(\lambda_3-\lambda_2)^2}\right)\cos 2\gamma, \tag{23}$$

where γ is the conjugate momentum of J_3 (relation (6)).

Equation (21) can also be used in (8′) in order to obtain the expression for $S=(\boldsymbol{u}_3\wedge\boldsymbol{v}).J$, and then D is obtained from relation (19) as

$$D=\left[\sum_{i=1}^{N}\left(\frac{J_2}{\lambda_3-\lambda_1}(r_{i_3}P_{i_1}+r_{i_1}P_{i_3})-\frac{J_1}{\lambda_3-\lambda_2}(r_{i_3}P_{i_2}+r_{i_2}P_{i_3})\right)-C(J^2-J_3^2)\right](J^2-J_3^2)^{-1/2}, \tag{24}$$

where $r_{i_j}=\boldsymbol{r}_i.\boldsymbol{u}_j$ and $P_{i_j}=\boldsymbol{P}_i.\boldsymbol{u}_j$.

It will be interesting to compare these results with those of the rigid body, since the picture of collective rotations of a system of particles is more or less connected with rigid body motion.

3. The rigid body

The canonical set (relation (6)) can be used to reduce the rigid body Hamiltonian. If the $\boldsymbol{u}_i$ are the principal directions of the inertial tensor, we have

$$H=J_1^2/2I_1+J_2^2/2I_2+J_3^2/2I_3. \tag{25}$$

The I_i are the principal momenta of inertia; using (6) we obtain

$$H=J_3^2/2I_3+\tfrac{1}{2}(J^2-J_3^2)(1/2I_1+1/2I_2)+\tfrac{1}{2}(J^2-J_3^2)(1/2I_2-1/2I_1)\cos 2\gamma. \tag{26}$$

It can be seen easily that another choice for the $\boldsymbol{u}_i$ would give a $(J^2-J_3^2)^{1/2}$ term in (26). In any case J_z, β and α are cyclic variables and do not appear in H. J is a constant of motion. The problem is then of one degree of freedom. The solution can be reached with the use of Jacobian elliptic functions, taking account of the conservation of the energy which gives a relation between J_3 and γ. In fact we again find known results (see Whittaker 1965), but in a more straightforward and, from a fundamental point of view, more attractive way.

We can now compare expression (26) with the decomposition (18), with C and D values given by (23) and (24). In (26) we have $D=0$, which is not the case in (24). The functions C in (26) and (23) are different but have the same structure; indeed

$$C(\text{rigid body})=\tfrac{1}{2}(1/2I_1+1/2I_2)+\tfrac{1}{2}\cos 2\gamma(1/2I_2-1/2I_1),$$

$$C(\text{particles})=\tfrac{1}{2}[I_2/(I_1-I_3)^2+I_1/(I_2-I_3)^2]+\tfrac{1}{2}\cos 2\gamma[I_2/(I_1-I_3)^2-I_1/(I_2-I_3)^2].$$

We have seen that a potential energy of the kind $V=\Sigma_{i>j} V_{ij}(|\boldsymbol{r}_i-\boldsymbol{r}_j|)$ does not change the values of C and D. Hence a rigid body must be seen as a system of particles with the potential energy depending not only on the positions but also on velocities. This potential energy is itself a function of J such that it will give the good rigid body result when added to the kinetic energy. This potential energy is defined simply by identification. We can write

$$V=\tfrac{1}{2}C'(J^2-J_3^2)+D'(J^2-J_3^2)^{1/2}+E', \tag{27}$$

with

$$C'=\tfrac{1}{2}(1/2I_1+1/2I_2)+\tfrac{1}{2}\cos 2\gamma(1/2I_2-1/2I_1)-C,$$
$$D'=-D, \qquad E'=J_3^2/2I_3-E,$$

where C, D, E are the quantities issuing from the kinetic energy expansion.

4. Conclusions

We have been able to give, in explicit form, the dependence on J of a large class of Hamiltonians. The comparison with the rigid body case led to interesting conclusions concerning the potential energy. In fact collective rotational motion may not be of the rigid body type. The potential energy (27) gives rigid body type rotational motion, but one can think of different possibilities which would cause other kinds of collective rotational motion.

Appendix

Let S be a scalar:

$$\{J_x, S\}=\{J_y, S\}=\{J_z, S\}=0.$$

We look for the expression of $\{\alpha, S\}$, where α is the conjugate momentum of J, as given by (6); we obtain

$$\{\alpha, S\}=\frac{(YJ_x-XJ_y)^2}{(YJ_x-XJ_y)^2+Z^2J^2}\left\{\frac{ZJ}{YJ_x-XJ_y}, S\right\}. \tag{A1}$$

X, Y, Z are the components of $\boldsymbol{\rho}$ given by (5); since $\boldsymbol{\rho}\,.\,\boldsymbol{J}=0$ and $\rho^2=(J^2-J_3^2)/J^2$, we find

$$(YJ_x-XJ_y)^2+Z^2J^2=\rho^2(J_x^2+J_y^2)=(J^2-J_3^2)(J_x^2+J_y^2)/J^2. \tag{A2}$$

The CPB in (A1) leads to

$$\left\{\frac{ZJ}{YJ_x-XJy}, S\right\}=\frac{J}{(YJ_x-XJ_y)^2}[\{Z, S\}(YJ_x-XJ_y)-ZJ_x\{Y, S\}+ZJ_y\{X, S\}].$$

Let us examine the bracket of the preceding relation. If we call $\boldsymbol{V}$ the vector $\boldsymbol{V}=\{\boldsymbol{\rho}, S\}$, we obtain

$$\left\{\frac{ZJ}{YJ_x-XJ_y}, S\right\}=\frac{J}{(YJ_x-XJ_y)^2}[\boldsymbol{\rho}\wedge \boldsymbol{V})_xJ_x+(\boldsymbol{\rho}\wedge \boldsymbol{V})_yJ_y]. \tag{A3}$$

But $\boldsymbol{\rho}$ is orthogonal to $\boldsymbol{J}$; then

$$\boldsymbol{J}\,.\,\boldsymbol{V}=\boldsymbol{J}\,.\,\{\boldsymbol{\rho}, S\}=\{\boldsymbol{J}\,.\,\boldsymbol{\rho}, S\}=0.$$

As the two vectors $\boldsymbol{\rho}$ and $\boldsymbol{V}$ are orthogonal to $\boldsymbol{J}$, it follows that $(\boldsymbol{\rho}\wedge\boldsymbol{V})$ is along $\boldsymbol{J}$; then we can write

$$\boldsymbol{\rho}\wedge\boldsymbol{V}=(\boldsymbol{J}/J^2)[(\boldsymbol{\rho}\wedge\boldsymbol{V})\,.\,\boldsymbol{J}]$$

and

$$(\boldsymbol{\rho}\wedge\boldsymbol{V})_x J_x+(\boldsymbol{\rho}\wedge\boldsymbol{V})_y J_y=[(J_x^2+J_y^2)/J^2][(\boldsymbol{\rho}\wedge\boldsymbol{V})\,.\,\boldsymbol{J}].$$

This last result, together with relation (A2), leads to

$$\{\alpha, S\}=[J/J^2-J_3^2)](\boldsymbol{\rho}\wedge\boldsymbol{V})\,.\,\boldsymbol{J}.$$

If now we let

$$\boldsymbol{v}=\{\boldsymbol{u}_3, S\} \qquad \text{(A4)}$$

then

$$\boldsymbol{V}=\boldsymbol{v}-(\boldsymbol{J}\,.\,\boldsymbol{v})\boldsymbol{J}/J^2.$$

Therefore, with expression (5) for ρ, we find

$$\{\alpha, S\}=[J/(J^2-J_3^2)](\boldsymbol{u}\wedge\boldsymbol{v})\,.\,\boldsymbol{J}. \qquad \text{(A5)}$$

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